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Ladder operators for subtle hidden shape-invariant potentials

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Abstract
Ladder operators can be constructed for all potentials that present the integrability condition known as shape invariance, satisfied by most of the exactly solvable potentials. Using the superalgebra of supersymmetric quantum mechanics, we construct the ladder operators for two exactly solvable potentials that present a subtle hidden shape invariance.

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1. Introduction

Two decades ago, it was shown that a subset of the exactly solvable potentials share an integrability condition characterized by a reparametrization invariance known as shape invariance [1]. In other words, not all the exactly solvable potentials seem to be shape invariant, a property introduced within the concept of supersymmetric quantum mechanics (SQM) [2]. Ten years ago, it was shown that this shape invariance condition has an underlying algebraic structure associated with Lie algebras [3–7]. For the potentials that share this property, it is possible to define coherent states and ladder operators defined in terms of the bosonic operators of the superalgebra, similar to the harmonic oscillator ladder operators.

Here our interest is in two particular exactly solvable potentials that, although known not to share this property, appear to have a subtle shape invariance, hidden by a special choice of the parameters of the transformation. In what follows, we introduce the general formulation to construct the ladder operators and then apply the methodology to two different exactly solvable potentials that do not present shape invariance at first, the case of the free particle confined in a box and the case of the Hulthén potential.

Consider a system described by a given potential V. The associated Hamiltonian H can be factorized in terms of bosonic operators and its lowest energy state, in $\hbar = c = 1$ units [2],

$$H_+ = H - E_0 = -\frac{d^2}{dx^2} + V_+(x) = A^+A^- \tag{1}$$

where E_0 is the lowest eigenvalue. The bosonic operators are defined in terms of the superpotential $W(x, a)$, which is a function of the position variable and a set of parameters, a, that represent space-independent properties of the original potential $V(r)$

$$A^{\pm} = \left(\mp\frac{d}{dx} + W(x, a)\right) \tag{2}$$

$$H_+ = -\frac{d^2}{dx^2} + W^2(x, a) - W'(x, a). \tag{3}$$

The partner Hamiltonian of H_- is given by

$$H_- = A^-A^+ = -\frac{d^2}{dx^2} + V_-(x) \tag{4}$$

$$H_- = -\frac{d^2}{dx^2} + W^2(x, a) + W'(x, a). \tag{5}$$

The Hamiltonians H_+ and H_- have the same spectra except for the ground state of H_+, for which there is no corresponding state in the spectra of H_-. As a consequence of the factorization of the Hamiltonian H, the Riccati equation must be satisfied,

$$W^2(x, a) - W'(x, a) = V(x) - E_0 = V_+ \tag{6}$$

and the corresponding potential $V_-(x)$ satisfies

$$W^2(x, a) + W'(x, a) = V_-(x) \tag{7}$$

The shape-invariant condition states that

$$V_-(x, a_0) - V_+(x, a_1) = R(a_1) \tag{8}$$

where $R(a_1)$ is independent of any dynamical variable and $a_1 = f(a_0)$. In terms of the bosonic operators the above condition is given by

$$A^-(a_0)A^+(a_0) - A^+(a_1)A^-(a_1) = R(a_1). \tag{9}$$

Potentials that satisfy this condition are exactly solvable. The contrary is not true: an exactly solvable potential may not be shape invariant. In this work, we consider the shape invariance involving translations of the parameters a:

$$a_1 = a_0 + \eta \tag{10}$$

where η is the translation step. Thus, it is possible to define operators $T(a_0)$ as

$$T(a_0) = \exp\left(\eta\frac{\partial}{\partial a_0}\right) \tag{11}$$

and

$$T^{-1}(a_0) = T^{\dagger}(a_0) = \exp\left(-\eta\frac{\partial}{\partial a_0}\right). \tag{12}$$

These operators act only on the objects defined in the parameters space.

Now we introduce the ladder operators, such as the creation and annihilation operators, by composing the translation operators T and the bosonic operators $A^{\pm}$,

$$B_{+}(a_0) = A^{+}(a_0)T(a_0) \qquad B_{-}(a_0) = T^{\dagger}(a_0)A^{-}(a_0). \tag{13}$$

The operators $B_{\pm}$ present the necessary algebraic structure [5] to identify them as ladder operators. As such they are analogous to the harmonic oscillator ladder operators

$$H_{+} = A^{+}A^{-} = B_{+}B_{-}. \tag{14}$$

Thus, the ground state must obey

$$B_{-}(a_0)\Psi_0(x, a_0) = A^{-}(a_0)\Psi_0(x, a_0) = 0 \tag{15}$$

or equivalently

$$\Psi_0(x, a_0) = N \exp\left(-\int_0^x W(\bar{x})\,\mathrm{d}\bar{x}\right). \tag{16}$$

The excited states are obtained by the repeated action of the creation operator on the ground state

$$\Psi_n(x, a_0) = (B_{+})^n(a_0)\Psi_0(x, a_0). \tag{17}$$

At this point, we emphasize that this algebraic approach is self-consistent and it allows us to determine the energy eigenvalues and eigenfunctions of a bound-state Schrödinger equation from the supersymmetric and shape invariance properties of the system. The energy is given by

$$E_n = E_0 + \sum_{k=1}^{n} R(a_k). \tag{18}$$

2. The free particle in a box

Consider the case of a free particle confined in a box of infinite walls. The potential is written as

$$\begin{aligned} V(x) &= 0 \qquad && 0 < x < \pi \\ &= \infty && -\infty < x < 0 \quad x > \pi \end{aligned} \tag{19}$$

and the factorized Hamiltonian in this case is [2]

$$\begin{aligned} H_{+} = H - E_0 &= -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_{+}(x) \\ &= -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 1 \end{aligned} \tag{20}$$

where H is the original Hamiltonian with the ground state energy eigenvalue $E_0 = -1$ so that the ground state of H_{+} is zero. The superpotential that factorizes H_{+} is

$$W(x) = -\cot(x) \tag{21}$$

and its supersymmetric partner is

$$\begin{aligned} H_{-} &= -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_{-}(x) \\ &= -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{2}{\sin^2(x)} - 1. \end{aligned} \tag{22}$$

At this point, we recollect the general form for the superpotential of the hierarchy [8]

$$W_n(x) = -n\cot(x) \tag{23}$$

where n is a natural number different from zero ($n = 1, 2, 3, \ldots$). The hierarchy is such that $E_n^{(1)} = n^2$ and the nth member of the super-family potential is

$$V_n(x) - E_0^{(n)} = \frac{n(n-1)}{\sin^2(x)} - n^2. \tag{24}$$

Thus, it is not shape invariant since $V_+ = -1$ and $V_- = \frac{2}{\sin^2(x)} - 1$. However, inspired by the superpotential of the hierarchy, equation (23), we rewrite the superpotential of H_+ in terms of a parameter a_0,

$$W(x, a_0) = -a_0\cot(x). \tag{25}$$

The superpotential (25) is a special case of the more general Infeld–Hull type E potential [9], whose shape invariance has been discussed in [3]. The related Hamiltonian is given by

$$\begin{aligned} H_+ &= -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_+ \\ &= -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{a_0(a_0-1)}{\sin^2(x)} - a_0^2. \end{aligned} \tag{26}$$

Its supersymmetric partner is given by

$$\begin{aligned} H_- &= -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_- \\ &= -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{a_0(a_0+1)}{\sin^2(x)} - a_0^2. \end{aligned} \tag{27}$$

Thus, setting $a_0 = 1$, we recover $H_\pm$ of the free particle given by equations (20) and (22). Now we can test the shape invariance. Substituting the potentials of equations (26) and (27) into (8), we obtain the following expression:

$$R(a_1) = \left(\frac{a_0(a_0+1)}{\sin^2(x)} - a_0^2\right) - \left(\frac{a_1(a_1-1)}{\sin^2(x)} - a_1^2\right) \tag{28}$$

which is an x-independent for $a_1 = a_0 + 1$. The step is then $\eta = 1$ and thus

$$R(a_1) = a_1^2 - a_0^2 = 2a_0 + 1. \tag{29}$$

The other steps shall be given by $a_k = a_0 + k$ and

$$\begin{aligned} R(a_k) &= a_k^2 - a_{k-1}^2 \\ &= (a_0 + k\eta)^2 - (a_0 + (k-1)\eta)^2 \\ &= 2k + 1 \end{aligned} \tag{30}$$

where we have set $a_0 = 1$. The energy levels, evaluated from equation (18), will be given by

$$\begin{aligned} E_n &= E_0 + \sum_{k=1}^{n} R(a_k) \\ &= 1 + \sum_{k=1}^{n} (2k+1) \\ &= (n+1)^2 \end{aligned} \tag{31}$$

as expected. The ladder operators, evaluated from equations (2) and (13) and the superpotential (25), are then given by

$$B_{+}(a_0) = \left(-\frac{\mathrm{d}}{\mathrm{d}x} - a_0 \cot(x)\right) \exp\left(-\frac{\partial}{\partial a_0}\right) \tag{32}$$

and

$$B_{-}(a_0) = \exp\left(-\frac{\partial}{\partial a_0}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} - a_0 \cot(x)\right) \tag{33}$$

and from the fact that

$$B_{-}(a_0)\Psi_0(x, a_0) = 0 \tag{34}$$

we arrive at

$$\Psi_0(x, a_0) \propto (\sin x)^{a_0} = \sin x \qquad a_0 = 1 \tag{35}$$

the ground state of the starting Hamiltonian. The excited states are constructed through the action of $B_{+}(a_0)$ in the ground state. For the first excited state, we have

$$\Psi_1(x, a_0) \propto B_{+}(a_0)\Psi(x, a_0) = -(2a_0 + 1)\cos x (\sin x)^{a_0} \tag{36}$$

and this gives

$$\Psi_1(x, a_0) \propto -\tfrac{3}{2} \sin 2x \qquad a_0 = 1 \tag{37}$$

which is correct apart from a normalization factor.

3. The Hulthén potential

The Hulthén potential, in atomic units, is given by

$$V_H(x) = -\frac{2\delta\, \mathrm{e}^{-\delta x}}{1 - \mathrm{e}^{-\delta x}} \tag{38}$$

where δ is the screening parameter. From early results [10], the partner Hamiltonians are given by

$$\begin{aligned} H_{+} = H - E_0 &= -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_{+} \\ &= -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{2\delta\, \mathrm{e}^{-\delta x}}{1 - \mathrm{e}^{-\delta x}} + \left(1 - \frac{\delta}{2}\right)^2 \end{aligned} \tag{39}$$

and

$$\begin{aligned} H_{-} &= -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_{-} \\ &= -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{2\delta\, \mathrm{e}^{-\delta x}}{1 - \mathrm{e}^{-\delta x}} + \frac{2\delta^2\, \mathrm{e}^{-\delta x}}{(1 - \mathrm{e}^{-\delta x})^2} + \left(1 - \frac{\delta}{2}\right)^2 \end{aligned} \tag{40}$$

where H is the Hamiltonian of the original problem and the superpotential is given by

$$W(x) = -\frac{\delta\, \mathrm{e}^{-\delta x}}{1 - \mathrm{e}^{-\delta x}} + 1 - \frac{\delta}{2}. \tag{41}$$

From equations (39) and (40), we see that V_+ and V_- surely are not shape invariant since we cannot satisfy equation (8) by changing parameters between them. At this point, however, we address ourselves to the results concerning the nth member of the Hulthén hierarchy [10]. The superpotential is given by

$$W_n(x) = -\frac{n\delta\,\mathrm{e}^{-\delta x}}{1-\mathrm{e}^{-\delta x}} + \frac{1}{n} - \frac{n}{2}\delta \tag{42}$$

which corresponds to the nth member of the Hamiltonian hierarchy with the potentials

$$\begin{aligned} V_n(x) - E_0^{(n)} &= W_n^2(x) - \frac{\mathrm{d}}{\mathrm{d}r}W_n(x) \\ &= \frac{n(n-1)\delta^2\,\mathrm{e}^{-\delta x}}{(1-\mathrm{e}^{-\delta x})^2} - \frac{2\delta\,\mathrm{e}^{-\delta x}}{1-\mathrm{e}^{-\delta x}} + \left(\frac{1}{n} - \frac{n}{2}\delta\right)^2. \end{aligned} \tag{43}$$

Note that V_+ corresponds to $n = 1$. Thus, again inspired by the hierarchy superpotential, we suggest writing the superpotential in terms of a quantity a_0 such that

$$W(x, a_0) = -\frac{a_0\delta\,\mathrm{e}^{-\delta r}}{1-\mathrm{e}^{-\delta r}} + \frac{1}{a_0} - \frac{a_0}{2}\delta \tag{44}$$

and evaluate the Hamiltonian

$$\begin{aligned} H_+ &= \left(-\frac{\mathrm{d}}{\mathrm{d}x} + W(x,a_0)\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} + W(x,a_0)\right) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_+(x,a_0) \\ &= -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{2\delta\,\mathrm{e}^{-\delta x}}{1-\mathrm{e}^{-\delta x}} + \frac{a_0(a_0-1)\delta^2\,\mathrm{e}^{-\delta x}}{(1-\mathrm{e}^{-\delta x})^2} + \left(\frac{1}{a_0} - \frac{a_0\delta}{2}\right)^2. \end{aligned} \tag{45}$$

Its supersymmetric partner is given by

$$\begin{aligned} H_- &= \left(\frac{\mathrm{d}}{\mathrm{d}x} + W(x,a_0)\right)\left(-\frac{\mathrm{d}}{\mathrm{d}x} + W(x,a_0)\right) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_-(x,a_0) \\ &= -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{2\delta\,\mathrm{e}^{-\delta x}}{1-\mathrm{e}^{-\delta x}} + \frac{a_0(a_0+1)\delta^2\,\mathrm{e}^{-\delta x}}{(1-\mathrm{e}^{-\delta x})^2} + \left(\frac{1}{a_0} - \frac{a_0\delta}{2}\right)^2. \end{aligned} \tag{46}$$

Thus, setting $a_0 = 1$, we recover H_+ and H_- of the original Hulthén problem given by equations (39) and (40).

Now we can test the shape invariance. Substituting the potentials of equations (45) and (46) into (8), we obtain an x-independent expression,

$$R(a_1) = \left(\frac{1}{a_0} - \frac{a_0\delta}{2}\right)^2 - \left(\frac{1}{a_1} - \frac{a_1\delta}{2}\right)^2 \tag{47}$$

for $a_1 = a_0 + 1$. The step $\eta = 1$. The other steps shall be given by $a_k = a_0 + k$. This gives

$$R(a_k) = \left(\frac{1}{a_{k-1}} - \frac{a_{k-1}\delta}{2}\right)^2 - \left(\frac{1}{a_k} - \frac{a_k\delta}{2}\right)^2 \tag{48}$$

which results in

$$R(a_k) = \frac{1+2k}{k^2(k^2+1)} - (2k+1)\frac{\delta^2}{4} \tag{49}$$

when we set $a_0 = 1$. The energy levels are then given by

$$E_n = E_0 + \sum_{k=1}^{n} R(a_k)$$
$$= -\left(\frac{1}{n+1} - (n+1)\frac{\delta}{2}\right)^2 \qquad n = 0, 1, 2, \ldots \tag{50}$$

as expected.

The ladder operators, evaluated from equations (2) and (13) and the superpotential (44), are then given by

$$B_+(a_0) = \left(-\frac{\mathrm{d}}{\mathrm{d}x} - \frac{a_0\delta\,\mathrm{e}^{-\delta x}}{1-\mathrm{e}^{-\delta x}} + \frac{1}{a_0} - \frac{a_0}{2}\delta\right)\exp\left(\frac{\partial}{\partial a_0}\right) \tag{51}$$

and

$$B_-(a_0) = \exp\left(-\frac{\partial}{\partial a_0}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{a_0\delta\,\mathrm{e}^{-\delta x}}{1-\mathrm{e}^{-\delta x}} + \frac{1}{a_0} - \frac{a_0}{2}\delta\right) \tag{52}$$

and from the fact that

$$B_-(a_0)\Psi_0(x, a_0) = 0 \tag{53}$$

we arrive at

$$\Psi_0(x, a_0) = \exp\left(-\left(\frac{1}{a_0} - \frac{a_0\delta}{2}\right)x\right)(1-\mathrm{e}^{-\delta x})^{a_0} \tag{54}$$

the ground state of the starting Hamiltonian. The excited states are constructed through the action of $B_+(a_0)$ in the ground state. The first excited state is given by

$$\Psi_1(x, a_0) \propto B_+(a_0)\Psi(x, a_0) = A_+(a_0)\Psi(x, a_0+1) \tag{55}$$

and this gives, for $a_0 = 1$,

$$\Psi_1(x) \propto \tfrac{3}{2}\exp\left(-\left(\tfrac{1}{2} - \delta\right)x\right)(1-\mathrm{e}^{-\delta x})((1-\delta) - \mathrm{e}^{-\delta x}(1+\delta)) \tag{56}$$

as expected.

4. Conclusions

We have shown that two different exactly solvable potentials present a hidden shape invariance, which is seen after implementing a parameter which will develop the required transformation. The implementation of this parameter, which is in fact fixed and equal to the unity in both cases, enables the construction of the ladder operators analogous to the creation and annihilation operator of the harmonic oscillator case.

We notice that Hulthén potential, written in terms of hyperbolic functions, is a particular case of the exactly solvable and shape-invariant Eckart potential for fixed values of the parameters. However this is not obvious at first, starting from the factorization of the Hulthén potential and testing the shape-invariance condition. The same argument is valid to the case of the free particle confined in a box and its relation to the Rosen–Morse I potential.

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